Chaos-Induced Diffusion.
Analogues to Nonlinear Fokker-Planck Equations

H. Fujisaka
Department of Physics, Kagoshima University, Kagoshima 890, Japan

S. Grossmann
Fachbereich Physik, Philipps-Universität, Renthof 6, D-3550 Marburg, FRG

S. Thomae
Institut für Festkörperforschung, Kernforschungsanlage Jülich, D-5170 Jülich, FRG

Deterministic chaotic dynamics induced by a climbing map with variable local jump size is shown to yield anomalous diffusion, i.e. a nonlinear increase of the variance with time. The relation to Fokker-Planck equations with variable drift and diffusivity is given. Turbulent two-particle diffusion may be a possible application.

§1. Introduction

Many investigations [1–7] have recently been devoted to the large-scale diffusive behavior of the solutions of chaotic maps. In [3] we have developed a general approach to the large-scale dynamics of maps as shown in Fig. 1, bottom,

$$Y_{t+1} = G(Y_t) = Y_t + \tilde{G}(Y_t), \quad -\infty < Y_t < \infty,$$
$$t = 0, 1, 2, \dots, \quad (1)$$

where $\tilde{G}(Y)$ is a 1-periodic function $\tilde{G}(Y+1) = \tilde{G}(Y)$. The periodicity of $\tilde{G}$ induces a decomposition of the Y-space into cells $[N, N+1)$ of unit length which was used in [3] to derive analytic results for the drift velocity v_0 and the diffusion constant D_0 of an ensemble of trajectories of G.

Because of the periodicity of $\tilde{G}$ the trajectories generated by G correspond to a chaotic motion in a strictly periodic potential. The phase dynamics of a periodically driven damped anharmonic oscillator (e.g. Josephson junction [8]) may serve as an example.

After a brief review of the results for homogeneous systems in §2 we introduce the conjugating function in §3 which leads to the description of deterministic diffusion in inhomogeneous systems. §4 contains examples of systems with a power-law and an exponential-law inhomogeneity. In §5 we show

how the conjugation concept may also be applied to randomly distorted homogeneous systems.

§2. Review of the results for the homogeneous case and generalization to varying cell-sizes

To introduce a generalization of (1) to varying cell-sizes we briefly review some of the results obtained in [3] for the homogeneous case.

The dynamics of (1) can be decomposed into a large-scale part N_t and a small-scale part y_t. N_t denotes the integer part of Y_t and y_t its fractional part:

$$Y_t = N_t + y_t, \quad N_t = [Y_t], \quad \text{largest integer} \leq Y_t,$$
$$y_t \in [0, 1). \quad (2)$$

Then (1) becomes

$$N_{t+1} = N_t + \Delta(y_t),$$
$$y_{t+1} = g(y_t), \quad (3)$$

with the jump function

$$\Delta(y) = [G(y)], \quad y \in [0, 1), \quad (4)$$

and the reduced map

$$g(y) = G(y) - \Delta(y), \quad y \in [0, 1). \quad (5)$$

The reduced map is assumed to be ergodic and mixing. Here there exists an absolutely continuous invariant density $\varrho(y)$ normalized on $[0, 1)$, which is
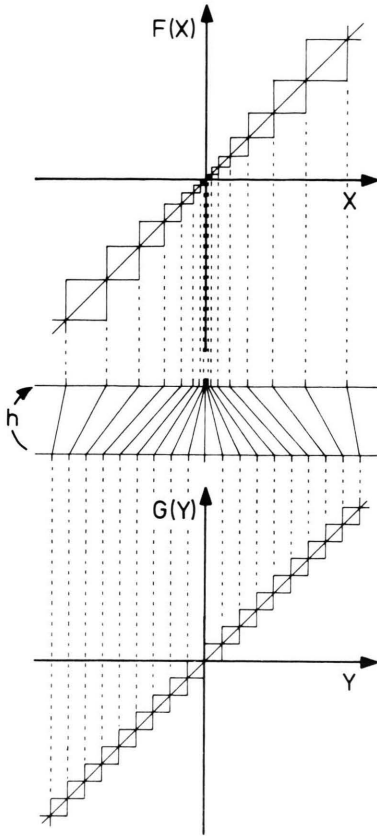
Fig. 1. Conjugation $X = h(Y)$ from the homogeneous lattice in Y-space to the distorted lattice in X-space. To indicate the distortion effected by the conjugating function h, the cells indicating the periodic structure of $G(Y)$ and the corresponding one of $F(X)$ are shown instead of the functions themselves. In this example h is proportional to Y^3 (cf. turbulent two-particle diffusion).

the unique solution of

$$\varrho(y) = \int_0^1 d\tilde{y} \, \delta(y - g(\tilde{y})) \, \varrho(\tilde{y}) . \qquad (6)$$

The average over all initial points $N_0 + y_0$ within a chosen initial cell N_0 with respect to this density will henceforth be denoted by $\langle \ldots \rangle$, e.g.

$$\langle Y_t \rangle = \int_0^1 dy_0 \, \varrho(y_0) \, Y_t(N_0 + y_0) . \qquad (7)$$

Since g is a chaotic map, the long-time behavior of the solutions $\{Y_t\}$ of (1) is well described by the drift

$$v_0 = \lim_{t \to \infty} \frac{1}{t} \langle Y_t - Y_0 \rangle = \langle \Delta \rangle \qquad (8)$$

and the diffusion constant

$$D_0 = \lim_{t \to \infty} \frac{1}{2t} \langle (Y_t - \langle Y_t \rangle)^2 \rangle . \qquad (9)$$

If the correlation between jumps $\Delta(y_t)$ decays sufficiently fast, $D_0 > 0$ is finite [3, 4], and the moments $\langle Y_t^m \rangle$ behave for large t like the moments of

$$P(Y, t) = \frac{1}{\sqrt{4\pi D_0 t}} \exp\left(-\frac{1}{2} \frac{(Y - Y_0 - v_0 t)^2}{2 D_0 t}\right),$$
$$t \to \infty , \qquad (10)$$

which is the solution of the Fokker-Planck equation

$$\partial_t P(Y, t) = -\partial_y (v_0 - \partial_y D_0) P(Y, t) . \qquad (11)$$

This means, one always finds asymptotically normal diffusion with a temporal increase of the variance proportional to t. The details of the reduced map only determine the magnitude of the diffusivity D_0. Before reaching the asymptotic time regime the variance increases as t^2 and continuously switches to normal behavior according to the correlation decay of $\Delta(y_t)$ [5]. This feature is well known from thermal Brownian motion. The "anomalous" exponent $\neq 1$ indicates correlation between successive jumps.

Note that the assumption of the existence of a properly defined invariant density ϱ excludes an anomalous diffusion caused by an intermittent reduced map. Such a case was described in [6]. There the trajectory could get "trapped" by an intermittency center in an interval where $\Delta(y) = 0$. This leads to an algebraically decaying waiting-time distribution and thus to an anomalous diffusion.

§3. Maps with varying cell-sizes

A generalization of (1) to maps with varying cell-sizes can be achieved by conjugation. If $[Q_N, Q_{N+1})$ is the N-th cell with a size

$$l_N = Q_{N+1} - Q_N > Q , \qquad (12)$$

we choose a continuous, piecewise differentiable, invertible function h with

$$Q_N = h(N) , \quad N = 0, \pm 1, \pm 2, \ldots . \qquad (13)$$

Any h with these properties will be called a conjugating function. $X = h(Y)$ maps the dynamical variable Y onto another dynamical variable X gov-

erned by the equation

$$X_{t+1} = F(X_t), \quad t = 0, 1, 2, \dots \tag{14}$$

with $F = h \circ G \circ h^{-1}$. h is a one-to-one mapping of the homogeneous system onto an inhomogeneous system with variable cell-size (Figure 1). The density $P(Y, t)$ on Y-space transforms into the X-space density

$$R(X, t) = P(h^{-1}(X), t) \left| \frac{dh^{-1}}{dX}(X) \right|. \tag{15}$$

The corresponding Fokker-Planck equation (11) transforms accordingly into

$$\partial_t R(X, t) = -\partial_X (v(X) - \partial_X D(X)) R(X, t) \tag{16}$$

with the space-dependent drift and diffusivity

$$v(X) = v_0 \sqrt{\frac{D(X)}{D_0}} + \frac{1}{2} \frac{dD}{dX}(X), \tag{17'}$$

$$D(X) = D_0 \left\{ \left. \frac{dh}{dY} \right|_{Y = h^{-1}(X)} \right\}^2. \tag{17''}$$

Thus, the conjugation method yields a class of chaos-induced diffusion processes equivalent to stochastic processes characterized via the general Fokker-Planck equation (16) by a space-dependent drift and diffusivity. If a dynamical system with *normal* diffusion is used as a starting point, $v(X)$ and $D(X)$ are related by (17). There is some analogy to the theory of stochastic processes: a Wiener-process in Y-space drives a more general process in X-space.

§4. Examples

An example for a deterministic diffusion process with varying drift and diffusivity is turbulent two-particle diffusion [9−11]. If X is interpreted as the interparticle distance, the diffusivity has to obey Richardson's 4/3-law [9] in the inertial subrange,

$$D(X) = D_0 |X|^{4/3}. \tag{18}$$

Furthermore, dimensional arguments (Kolmogorov, Oboukhov, v. Weizsäcker, Heisenberg, Onsager; cf. [10, 12]) imply $v(X) \propto |X|^{1/3}$. Hence $v_0 = 0$ in (17'). Integration of (17'') yields the conjugating function

$$h(Y) = \frac{1}{27} Y^3. \tag{19}$$

This example motivates the general study of a class of functions $h_\alpha(Y)$ with power-law behavior. For simplicity we confine ourselves to h_α with odd symmetry

$$h_\alpha(-Y) = -h_\alpha(Y). \tag{20}$$

We discuss in particular

$$h_\alpha(Y) = \begin{cases} Y |Y|^{\alpha-1} & \text{if } \alpha > 0, \\ \operatorname{sgn}(Y) \ln(1 + Y^2) & \text{if } \alpha = 0, \\ \operatorname{sgn}(Y) \{1 - (1 + Y^2)^{\alpha/2}\} & \text{if } \alpha < 0. \end{cases} \tag{21}$$

h_α generates maps F where the size l_N of cells behaves according to

$$l_N \propto |N|^{\alpha-1} \quad \text{for} \quad N \to \pm\infty. \tag{22}$$

If $\alpha < 0$, l_N decreases so rapidly that the totality of cells only fills a finite interval, which has been normalized to $[-1, 1]$ in (21). Table 1 shows the transport coefficients $v_\alpha(X)$ and $D_\alpha(X)$ for the diffusion in X-space, transformed via $h_\alpha(Y)$ from the homogeneous Wiener process. The asymptotic $(t \to \infty)$ behavior of the mean $m_\alpha(t) = \langle X \rangle_t$ and the variance $\sigma_\alpha(t) = \langle X^2 \rangle_t - \langle X \rangle_t^2$ can be evaluated in a lengthy and tedious but well-defined way. We only

Table 1. X-dependence of drift $v_\alpha(X)$ and diffusivity $D_\alpha(X)$ for the power-law conjugation (21). For $\alpha < 0$ the abbreviation $Z = 1 - |X|$ is used.

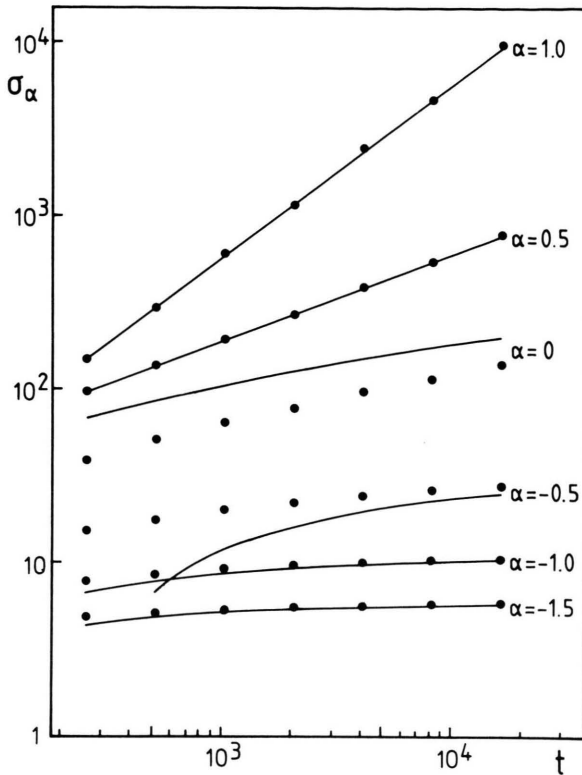| $h_\alpha(Y)$		$v_\alpha(X)$	$D_\alpha(X)$
$Y |Y|^{\alpha-1}$,	$\alpha > 0$	$\alpha \{v_0 + (\alpha - 1) D_0 \operatorname{sgn}(X) |X|^{-1/\alpha}\} |X|^{1-1/\alpha}$	$\alpha^2 D_0 |X|^{2-2/\alpha}$
$\operatorname{sgn}(Y) \ln(1 + Y^2)$,	$\alpha = 0$	$2 \left\{ v_0 - D_0 \operatorname{sgn}(X) \dfrac{e^{-|X|/2} - 2 e^{-3|X|/2}}{[1 - e^{-|X|}]^{1/2}} \right\}$ $\cdot \{e^{-|X|} - e^{-2|X|}\}^{1/2}$	$4 D_0 \{e^{-|X|} - e^{-2|X|}\}$
$\operatorname{sgn}(Y) \{1 - (1 + Y^2)^{\alpha/2}\}$,	$\alpha < 0$	$|\alpha| \left\{ v_0 - (1-\alpha) D_0 \operatorname{sgn}(X) \dfrac{Z^{-1/\alpha} - \dfrac{2-\alpha}{1-\alpha} Z^{-3/\alpha}}{[1 - Z^{-2/\alpha}]^{1/2}} \right\}$ $\cdot \{Z^{2-2/\alpha} - Z^{2-4/\alpha}\}^{1/2}$	$\alpha^2 D_0 \{Z^{2-2/\alpha} - Z^{2-4/\alpha}\}$

Fig. 2. Variances σ_α vs. time t for various α. The data points represent the numerically computed σ_α of an ensemble of 5000 points which were started in the first cell at $t = 0$. The lines indicate the behavior to be expected from the asymptotic formulae in Table 2. Note the slow convergence of data points to the asymptotic behavior near the localization transition $\alpha = 0$. The results for $\alpha = 0.5$ have been multiplied by a factor of 10 for clarity of representation.
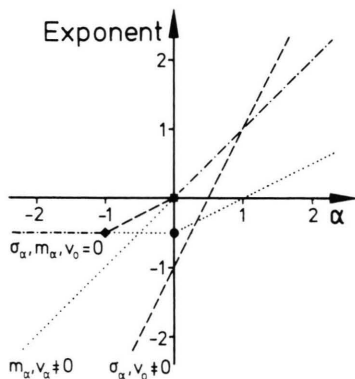

Fig. 3. The plot shows the exponent of t in the leading term as a function of α for power-law conjugation. The dotted lines refer to m_α, the dashed lines to σ_α. The symbols indicate additional factors of $\ln t$ in m_α (●,■) and σ_α (◆) as well as $(\ln t)^2$ in σ_α (■).

report the results in Table 2 and compare with numerical data in Figure 2.

If $\alpha = 1$ we recover normal diffusion. Other positive α yield anomalous diffusion. For negative α the dominating feature is the localization.

One important aspect of the results presented in Table 2 is the behavior of the exponents of t in the leading-order t-dependent terms of m_α and σ_α when α is changed. As one can see (Fig. 3), exponents are bounded below by $-1/2$ if $v_0 = 0$. The transition from the finite system ($\alpha < 0$) to the infinite system ($\alpha > 0$) appears as a critical situation where m_α as well as σ_α contain logarithmic corrections. σ_α undergoes another transition at $\alpha = -1$, which indicates the following qualitative change in the dynamics: For $\alpha > -1$ the increase of σ_α with time is still a consequence of a spreading over more and more cells. For $\alpha < -1$ the size of cells decreases so rapidly that the ensemble is spread over essentially all of the interval $[-1, 1]$ after a few time-steps. The further change of σ_α is then brought about by a decrease of the probability in the core of $[-1, 1]$ due to the normalization prefactor $\propto t^{-1/2}$ of $P(Y, t)$. If $v_0 \neq 0$, exponents are not bounded below. The drift "squeezes" the diffusing ensemble against the boundary of the system. The smaller $\alpha < 0$ becomes, the faster do the cells decrease in size when the boundary is approached. Therefore, the width σ_α of the ensemble shrinks and m_α approaches ± 1 the faster, the smaller α is. Again m_α has a logarithmic correction for $\alpha = 0$. However, σ_α does not. This can be explained as well by the squeezing effect. While the mean-motion really undergoes a qualitative change when the finite system is replaced by the infinite one (there are no longer any boundaries to stop it), the width does not experience the limitation due to finiteness of space since it is shrinking anyhow, because the ensemble is squeezed into smaller and smaller cells. This is already present for positive $\alpha < 1/2$.

Though for large $\alpha > 0$ σ_α increases with an arbitrarily high power of t, $D(X) \propto X^{2-2/\alpha}$ always increases more slowly than X^2 for $X \to \infty$. To have $D(X) \propto X^2$ requires an exponentially changing h_α. So we consider as a second example

$$h_\alpha(Y) = \begin{cases} \sinh(\alpha Y), & \alpha > 0, \\ \mathrm{sgn}(Y)\{1 - e^{\alpha|Y|}\}, & \alpha < 0. \end{cases} \quad (23)$$

The results for v_α, D_α, m_α, and σ_α are given in Tables 3 and 4, respectively. Because of the symmetry (20)

Table 2. Asymptotic time-dependence of the mean $m_\alpha(t)$ and the variance $\sigma_\alpha(t)$ for the power-law conjugation (21).

$v_0 = 0$:

α	$m_\alpha(t), \quad t \to \infty$	$\sigma_\alpha(t), \quad t \to \infty$
$\alpha > 0$	$\Gamma\left(1 + \dfrac{\alpha}{2}\right) \dfrac{2}{\sqrt{\pi}} Y_0 (4 D_0 t)^{(\alpha - 1)/2}$	$\Gamma\left(\dfrac{1}{2} + \alpha\right) \dfrac{1}{\sqrt{\pi}} (4 D_0 t)^\alpha$
$\alpha = 0$	$\dfrac{2}{\sqrt{\pi}} Y_0 (\ln(4 D_0 t) - \gamma)(4 D_0 t)^{-1/2}$	$[\ln(4 D_0 t)]^2$
$-1 < \alpha < 0$		$1 - \Gamma\left(\dfrac{1 + \alpha}{2}\right) \dfrac{2}{\sqrt{\pi}} (4 D_0 t)^{\alpha/2}$
$\alpha = -1$	$\dfrac{2}{\sqrt{\pi}} Y_0 (4 D_0 t)^{-1/2}$	$1 - \dfrac{2}{\sqrt{\pi}} \left\{\ln(4 D_0 t) - 2\ln 2 + \gamma + \dfrac{\pi}{2}\right\}(4 D_0 t)^{-1/2}$
$\alpha < -1$		$1 - \left\{2 \dfrac{\Gamma\left(-\dfrac{1 + \alpha}{2}\right)}{\Gamma\left(-\dfrac{\alpha}{2}\right)} - \dfrac{\Gamma\left(-\dfrac{1 + 2\alpha}{2}\right)}{\Gamma(-\alpha)}\right\}(4 D_0 t)^{-1/2}$

$\gamma = 0.577215664\ldots$ Euler's constant

$v_0 > 0$:

α	$m_\alpha(t), \quad t \to \infty$	$\sigma_\alpha(t), \quad t \to \infty$
$\alpha > 0$	$(v_0 t)^\alpha$	$\dfrac{\alpha^2}{2} (v_0 t)^{2(\alpha - 1)} (4 D_0 t)$
$\alpha = 0$	$2 \ln(v_0 t)$	$2 (v_0 t)^{-2} (4 D_0 t)$
$\alpha < 0$	$1 - (v_0 t)^\alpha$	$\dfrac{\alpha^2}{2} (v_0 t)^{2(\alpha - 1)} (4 D_0 t)$

Table 3. X-dependence of drift and diffusivity for the exponential conjugation (23).

$h_\alpha(Y)$		$v_\alpha(X)$	$D_\alpha(X)$
$\sinh(\alpha Y)$,	$\alpha > 0$	$\alpha\left\{v_0 + \alpha D_0 \dfrac{X}{\sqrt{1 + X^2}}\right\}\sqrt{1 + X^2}$	$\alpha^2 D_0 \{1 + X^2\}$
$\mathrm{sgn}(Y)\{1 - e^{\alpha\|Y\|}\}$,	$\alpha < 0$	$\|\alpha\| \{v_0 - \|\alpha\| D_0 \,\mathrm{sgn}(X)\}\{1 - \|X\|\}$	$\alpha^2 D_0 \{1 - \|X\|\}^2$

the mean m_α and the variance σ_α have the symmetry

$$m_\alpha(Y_0, v_0) = -m_\alpha(-Y_0, -v_0),$$
$$\sigma_\alpha(Y_0, v_0) = \sigma_\alpha(-Y_0, -v_0). \tag{24}$$

Therefore the results for $v_0 \gtreqless 0$ given in Tabs. 2 and 4 can easily be converted to $v_0 \lesseqgtr 0$.

The dynamical system defined by h_α in the X-space has two typical "length" scales, an inner scale l_0, which is the size of the smallest (largest) cell if $l_N \to \infty$ ($l_N \to 0$) for $|N| \to \infty$ and the total range

of diffusion L, which is finite for $\alpha < 0$. Introducing parameters $p, q > 0$ one can adjust the inner and outer scales to their proper values by using

$$\tilde{h}_\alpha(Y) = p\, h_\alpha(q Y). \tag{25}$$

The corresponding moments $\tilde{m}_\alpha$ and $\tilde{\sigma}_\alpha$ are related to those given in the tables by

$$\tilde{m}_\alpha(t; Y_0, v_0, D_0) = p\, m_\alpha(t; q Y_0, q v_0, q^2 D_0),$$
$$\tilde{\sigma}_\alpha(t; Y_0, v_0, D_0) = p^2 \sigma_\alpha(t; q Y_0, q v_0, q^2 D_0). \tag{26}$$

Table 4. Time-dependence of the mean $m_\alpha(t)$ and the variance $\sigma_\alpha(t)$ for the exponential conjugation (23). For $\alpha > 0$ the complete expressions are given, for $\alpha < 0$ asymptotically leading terms, only.

$\alpha > 0$: $\qquad m_\alpha(t) = \sinh\{\alpha(Y_0 + v_0 t)\}\, e^{\alpha^2 D_0 t}$ $\qquad \sigma_\alpha(t) = \frac{1}{2}\{e^{2\alpha^2 D_0 t} - 1\}$

$\alpha < 0$: $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\{\cosh[2\alpha(Y_0 + v_0 t)]\, e^{2\alpha^2 D_0 t} + 1\}$

v_0	$m_\alpha(t)$	$\sigma_\alpha(t)$
$v_0 = 0$	$2Y_0(4\pi D_0 t)^{-1/2}$	$1 - \dfrac{3}{|\alpha|}(4\pi D_0 t)^{-1/2}$
$v_0 = 2|\alpha|D_0$	$1 - \left\{\dfrac{1}{2} + \left(Y_0 + 3\dfrac{D_0}{v_0}\right)(4\pi D_0 t)^{-1/2}\right\}$ $\cdot\exp\left\{-\dfrac{Y_0 v_0}{2D_0}\right\}\exp\left\{-\dfrac{v_0^2 t}{4D_0}\right\}$	$\dfrac{20}{3}\dfrac{D_0}{v_0}(4\pi D_0 t)^{-1/2}$ $\cdot\exp\left\{-\dfrac{Y_0 v_0}{2D_0}\right\}\exp\left\{-\dfrac{v_0^2 t}{4D_0}\right\}$
$v_0 = 4|\alpha|D_0$	$1 - \exp\{\alpha Y_0 + \alpha(v_0 + \alpha D_0)t\}$	$\left\{\dfrac{1}{2} + \left(Y_0 + \dfrac{11}{3}\dfrac{D_0}{v_0}\right)(4\pi D_0 t)^{-1/2}\right\}$ $\cdot\exp\left\{-\dfrac{Y_0 v_0}{2D_0}\right\}\exp\left\{-\dfrac{v_0^2 t}{4D_0}\right\}$
$0 < v_0 < 2|\alpha|D_0$	$1 - \dfrac{16\alpha^2 D_0^2}{4\alpha^2 D_0^2 - v_0^2}\left(\dfrac{D_0}{v_0}\right)(4\pi D_0 t)^{-1/2}$ $\cdot\exp\left\{-\dfrac{Y_0 v_0}{2D_0}\right\}\exp\left\{-\dfrac{v_0^2 t}{4D_0}\right\}$	$\left[16|\alpha|\left\{\dfrac{v_0^2}{16\alpha^2 D_0^2 - v_0^2} - \dfrac{v_0}{2\alpha D_0 - v_0}\right\}\right.$ $\cdot\dfrac{D_0^2}{v_0^2}(4\pi D_0 t)^{-1/2}$ $\left.\cdot\exp\left\{-\dfrac{Y_0 v_0}{2D_0}\right\}\exp\left\{-\dfrac{v_0^2 t}{4D_0}\right\}\right.$
$\left.\begin{array}{l} 2|\alpha|D_0 < v_0 < (4+2\sqrt{2})|\alpha|D_0 \\ v_0 \neq 4|\alpha|D_0 \end{array}\right\}$	$1 - \exp\{\alpha Y_0 + \alpha(v_0 + \alpha D_0)t\}$	
$(4+2\sqrt{2})|\alpha|D_0 \leq v_0$		$\exp\{2\alpha Y_0 + 2\alpha(v_0 + 2\alpha D_0)t\}$

The attentive reader will have noticed that we have specified the conjugating function in §3 on the cell boundaries only. In between, h_α is arbitrary up to the restrictions of continuity and monotonicity. In fact, the details of h_α determine the fine structure of the probability density within the cells. But they do not have an influence upon the integral weight of each cell. The results obtained with the particular h_α specified in (21) and (23) thus also hold for h_α with different fine-structure if changes inside cells are negligible with respect to the overall features, i.e. $l_0 \ll L$.

§5. Random spatial fluctuations

The conjugation method can also be applied when the cell-sizes l_N, $N = 0, \pm 1, \pm 2, \ldots$ are random variables. To demonstrate this we consider a particular example:

We assume the l_N, $N = 0, \pm 1, \pm 2, \ldots$ to be identically distributed variables with a mean $\bar{l}$. The fluctuations $\delta l_N = l_N - \bar{l}$ are supposed to be mutually independent, i.e. $\overline{\delta l_N \delta l_{N'}} = \overline{(\delta l)^2}\,\delta_{NN'}$ with a finite variance $\overline{(\delta l)^2} < \infty$. The overbar denotes averaging with respect to the cell-size statistics. Introducing the notation

$$\sum_n^N \ldots = \begin{cases} \displaystyle\sum_{n=0}^{N-1}\ldots & \text{if } N > 0, \\[2mm] 0 & \text{if } N = 0, \\[2mm] -\displaystyle\sum_{n=N}^{-1}\ldots & \text{if } N < 0, \end{cases} \qquad (27)$$

we define a stochastic conjugating function by

$$h(N) = \sum_n^N l_n = N\bar{l} + \sum_n^N \delta l_n. \qquad (28)$$

To calculate $m(t)$ and $\sigma(t)$ we use a decomposition of $X = h(Y)$ based on the Y-decomposition (2)

$$X_t = h(N_t) + l_{N_t}\varkappa_t \quad \text{with} \quad \varkappa_t \in [0,1). \qquad (29)$$

Thus, one finds up to order $O(t^0)$ for the mean $m(t) = \overline{\langle X \rangle_t}$

$$m(t) = \bar{l} \overline{\langle N \rangle_t} = \bar{l} v_0 t. \qquad (30)$$

The leading order for the variance $\sigma(t) = \overline{\langle X^2 \rangle_t} - \overline{\langle X \rangle_t^2}$ reads

$$\sigma(t) = \bar{l}^2 (\overline{\langle N^2 \rangle_t} - \overline{\langle N \rangle_t^2}) + \overline{(\delta l)^2} \, \overline{\langle |N| \rangle_t} + O(t^0)$$

$$= \begin{cases} (2\bar{l}^2 D_0 + \overline{(\delta l)^2} \, |v_0|) \, t + O(t^0) & \text{if } v_0 \neq 0, \\[2mm] 2\bar{l}^2 D_0 t + \overline{(\delta l)^2} \, \sqrt{\dfrac{4 D_0}{\pi}} \, t^{1/2} + O(t^0) & \text{if } v_0 = 0, \end{cases} \qquad (31)$$

Note, the disorder in cell-sizes does not cause anomalous diffusion for $t \to \infty$ if, as assumed here, the cell-size fluctuations have a finite variance. The anomalous $t^{1/2}$-term dominates for small t only, i.e. if $\sqrt{\pi D_0 t} < \overline{(\delta l)^2} / \bar{l}^2$. If, however, the cell-size vari-

ance is infinite, one finds truly anomalous diffusion, as a proper discussion of (31) proves.

Since any such disordered system is topologically conjugate to the homogeneous Y-system by (28), the disorder is in the cell-sizes only. The cell-to-cell transition rates do not fluctuate. Therefore, these systems with random disorder correspond rather to stochastic systems with anomalous diffusion as described in [13] (fluctuations of lattice constant) then to those with randomly disordered transition rates (see e.g. [14]).

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